1. *Let $\{b_n\}_{n\geq 1}$ be a sequence defined by $b_1 = -20$ and*
$$b_{n+1} = 1 - \sqrt{1-b_n} \quad \text{for} \quad n \geq 1.$$
Show that the sequence $\{b_n\}_{n\geq 1}$ is convergent. Find its limit.

   Solution: If $b_k < 0$ for some k, then $\sqrt{1-b_k} > 1$. That is, $b_{k+1} = 1 - \sqrt{1-b_k} < 0$. Thus by mathematical induction principle, $b_n < 0$ for all n (since $b_1 < 0$). Since $1 - b_n > 1$, we have $\sqrt{(1-b_n)} \leq (1-b_n)$ for all n. Therefore, $b_{n+1} - b_n = (1-b_n) - \sqrt{1-b_n} \geq 0$. i.e., $\{b_n\}_{n\geq 1}$ is an increasing sequence. Also, we have observe that $\{b_n\}$ is bounded above by 0. Hence $\{b_n\}$ is a convergent sequence. Let b be the limit of $\{b_n\}$. Passing limit on both sides of the equation $b_{n+1} = 1 - \sqrt{1-b_n}$, we get that $1 - b = \sqrt{1-b}$. Therefore, $b = 0$ or 1. Since $b_n < 0$ for all n, $b \neq 1$. Hence, $\lim_{n\to\infty} b_n = 0$. □

2. *Let $a < b$ and $c < d$ be real numbers. Suppose $f : [a,b] \to [c,d]$ is a continuous bijection. Show that f^{-1} is continuous.*

   Solution: Since $[a,b]$ is a compact set, the result follows from Theorem 4.17 of Priniciples of Mathematical Analysis by Walter Rudin. □

3. *Let $\{x_n\}_{n\geq 1}$ be a bounded sequence of real numbers and let*
$$M = \limsup_{n\to\infty} x_n.$$
Show that there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that
$$\lim_{k\to\infty} x_{n_k} = M.$$

   Solution: See Theorem 3.17(a) of Priniciples of Mathematical Analysis by Walter Rudin. □

4. *Let $h : [-10, 10] \to \mathbb{R}$ be a differentiable function satisfying $h(-10) = -10$ and $h(10) = 10$. Suppose $h'(x) \leq 1$ for all $x \in [-10, 10]$. Show that $h(x) = x$ for all x.*

   Solution: Consider the function $g(x) = x - h(x)$. Then, $g'(x) = 1 - h'(x) \geq 0$ on $[-10, 10]$. Therefore g is an increasing function. But $g(10) = 0 = g(-10)$. Hence, $g(x) = 0$ for all x. That is, $h(x) = x$ for all x. □

5. *State and prove Rolle's theorem.*

   Solution: See Theorem 6.2.3 of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert. □

6. *Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be a function. Then f is said to satisfy a* **Lipschitz condition** *if there is a positive real number K such that*
$$|f(x) - f(y)| \leq K|x - y|$$
for all $x, y \in [a,b]$.

(i) *Give an example of a function not satisfying a Lipschitz condition.*

Solution: The function $f(x) = \sqrt{x}$ on $[0, 2]$ is not a Lipschitz function. See Example 5.46 (b) of Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert for more detail.

(ii) *Suppose* $f : [a, b] \to \mathbb{R}$ *is differentiable and the derivative* f' *is continuous on* $[a, b]$. *Show that* f *satisfies a Lipschitz condition.*

Solution: Since f' is continuous on a compact set $[a, b]$, there exist $M > 0$ such that $|f'(x)| \leq M$ for all $x \in [a, b]$. For $x, y \in [a, b]$ with $x < y$, by Mean Value Theorem, we have

$$f(x) - f(y) = f'(z)\,(y - x) \text{ for some } \ z \in (x, y).$$

Thus, $|f(x) - f(y)| = |f'(z)|\,|x - y| \leq M\,|x - y|$ for all $x, y \in [a, b]$. Hence, f is a Lipschitz function.

$\square$

7. *Show that if a series of real numbers* $\sum_{n=1}^{\infty} a_n$ *converges absolutely then* $\sum_{n=1}^{\infty} a_n^2$ *converges absolutely. Show that the converse is not true in general.*

Solution: Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then $a_n \to 0$ as $n \to \infty$. Choose N such that $|a_n| < 1$ for all $n > N$. Therefore, $|a_n|^2 \leq |a_n|$ for all $n > N$. Then by Comparision test, $\sum\limits_{n=N}^{\infty} |a_n|^2$ converges and hence $\sum a_n^2$ converges.

Converse of this result is not true in general. It can be seen by taking $a_n = 1/n$. Note that $\sum 1/n^p$ converges if and only if $p > 1$ (see Theorem 3.28 of Priniciples of Mathematical Analysis by Walter Rudin). $\square$